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On the covariant conservation law for non-Abelian external current

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Abstract. The SU(2)-covariant conservation law $D_\mu^{ba}J_\mu^a=0$ is discussed for various types of Euclidean non-vanishing external current J_μ^a. The classification of J_μ^a is performed in the gauge- and Euclidean-invariant way. It is shown that the intrinsically non-Abelian current can be arbitrary—no restriction is imposed on it by an interaction with a gauge field system. However the Abelian current has to be conserved. A decomposition of the gauge field potential A_μ^a is also performed in order to extract dynamical degrees of freedom from the non-dynamical ones.

1. Introduction

Non-Abelian gauge field theory with external currents introduces significant difficulties either on the classical or the quantum level. Recent papers on quantisation of this theory (Cabo and Shabad 1986, Przeszowski 1988) suggest that a basic change in our intuition is required. Here we would like to deal with a new interpretation of the so-called covariant conservation law for external current. We will consider the SU(2) gauge group and four-dimensional Euclidean spacetime. We suppose that the external current J_μ^a is real valued and arbitrary.

The classical dynamics is given by a non-homogeneous Yang–Mills equation

$$D_\mu^{ab}F_{\mu\nu}^b=J_\nu^a \tag{1}$$

where

$$F_{\mu\nu}^a=\partial_\mu A_\nu^a-\partial_\nu A_\mu^a+g\varepsilon^{abc}A_\mu^bA_\nu^c$$

$$D_\mu^{ab}=\delta^{ab}\partial_\mu+g\varepsilon^{acb}A_\mu^c.$$

The above system of partial differential equations has a non-trivial consistency condition

$$D_\mu^{ba}J_\mu^a=0 \tag{2}$$

which is usually known as the covariant conservation law for external current J_μ^a. This name is misleading because it suggests that there is some restriction to be put on J_μ^a in order to couple it unambiguously to a non-Abelian field A_μ^a (for example, Arodź 1983, Lai and Oh 1984, German 1984). Such a limitation really takes place either when $g\to 0$ or in an Abelian case where instead of (2) one has

$$\partial_\mu J_\mu^a=0 \qquad \text{or} \qquad \partial_\mu J_\mu=0 \tag{3}$$

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and there is no doubt that external current must be conserved. However, in the truly non-Abelian case one must remember that a covariant derivative contains a gauge field A^a_μ and one should not treat (2) perturbatively (in g). In this paper we show how equation (2) may be reinterpreted as *a constraint on gauge field degrees of freedom.* Thus we will conclude that an almost arbitrary external current J^a_μ may be coupled to a non-Abelian gauge field system in a consistent way.

Our method of presentation is as follows. First we find that all non-zero external currents fall into one of three possible classes. Every class is characterised in a gauge- and Euclidean-invariant way. Then we transform an invariant description of a class into some properties of any external current which belongs to this class. This deals with classification of external currents. Next we parametrise an arbitrary gauge field potential A^a_μ in such a way that some new degrees of freedom are constrained by the covariant conservation law (2). Every class of J^a_μ is discussed separately and for clarity most calculational details are omitted in the main presentation; they are given in the appendix.

2. Generic external current

Our equation of interest may be rewritten as

$$\partial_\mu J^a_\mu + g\varepsilon^{abc} A^b_\mu J^a_\mu = 0. \tag{2'}$$

Thus if we focus our attention on the second truly non-Abelian term then our problem becomes algebraic and local. Thus we will not write down explicitly the spacetime dependence of J^a_μ and A^a_μ, but we will presuppose that our analysis is to be carried out separately at every point. Furthermore, if the external current is smooth enough so that all final formulae exist everywhere, then our solution will become global.

We would like to discuss various cases of possible non-Abelian external current J^a_μ in a gauge- and Euclidean-covariant way: thus the following symmetric matrices will appear frequently:

$$K^{ab} = J^a_\mu J^b_\mu \tag{4a}$$

$$L_{\mu\nu} = J^a_\mu J^a_\nu. \tag{4b}$$

Though matrices $\mathbb{K}$ and $\mathbb{L}$ have different dimensions they have the same non-zero eigenvalues: thus in the SU(2) case in four dimensions we have for every J^a_μ

$$\mathrm{Det}\,\mathbb{L} \equiv 0 \tag{5}$$

(compare equation (A3) in the appendix). However, the other matrix $\mathbb{K}$ may have any number of non-zero eigenvalues. We will specify types of non-Abelian external current J^a_μ by the number of non-zero eigenvalues of $\mathbb{K}$.

First we will consider a generic case when

$$\mathrm{Det}\,\mathbb{K} \neq 0. \tag{6a}$$

Then $\mathbb{L}$ has only one zero eigenvalue or equivalently there is a non-zero eigenvector n_μ:

$$n_\mu L_{\mu\nu} = L_{\mu\nu} n_\nu = 0 \qquad n^2 = n_\mu n_\mu > 0. \tag{6b}$$

Further, from (6b), one obtains

$$n_\mu L_{\mu\nu} J^a_\nu = n_\mu J^b_\mu K^{ba} \equiv 0$$

and according to $(6a)$ one concludes that

$$n_\mu J^a_\mu \equiv 0. \tag{6c}$$

However, one should not confuse this property $(6c)$ of arbitrary J^a_μ with any constraint on it. *So far there is no restriction put on external current.* Equation $(6c)$ just says that any four three-dimensional (gauge) vectors must be linearly dependent.

Now we will solve equation (2) by pointing out which part of A^a_μ is constrained. We notice that gauge field A^a_μ can be decomposed as

$$A^a_\mu = \alpha^a n_\mu + D^T_{1\mu\nu}(S^{ab} + \varepsilon^{abc}N^c)J^b_\nu \tag{7a}$$

where

$$D^T_{1\mu\nu}L_{\nu\rho} = L_{\mu\nu}D^T_{1\nu\rho} = \delta_{\mu\rho} - n_\mu n_\rho / n^2 \tag{7b}$$

$$S^{ab} = \tfrac{1}{2}(J^a_\mu A^b_\mu + J^b_\mu A^a_\mu) \tag{7c}$$

$$\varepsilon^{abc}N^c = \tfrac{1}{2}(J^b_\mu A^a_\mu - J^a_\mu A^b_\mu) \tag{7d}$$

$$\alpha^a = A^a_\mu n_\mu / n^2 \tag{7e}$$

(for details see the appendix). Thus instead of A^a_μ we may treat α^a, S^{ab}, N^a as independent gauge field degrees of freedom. One should remember that the external current J^a_μ is a previously given quantity!

Accordingly our consistency condition (2) may be solved explicitly:

$$N^a = -(1/2g)\partial_\mu J^a_\mu \tag{8}$$

and our conclusion is that the covariant conservation law for non-Abelian external current (J^a_μ) gives a constraint on gauge field degrees of freedom (N^a) if this current is generic, e.g. if condition $(6a)$ is satisfied.

3. Non-generic external currents

Now we will discuss cases of degenerate external currents. Let us call J^a_μ non-generic if condition $(6a)$ is not satisfied, e.g. when

$$\mathrm{Det}\,\mathbb{K} = 0. \tag{9}$$

Because we are interested in non-zero J^a_μ, from (9) we conclude that the matrix $\mathbb{K}$ may have either one or two zero eigenvalues. These two possibilities will be discussed in sequence. For clarity we will omit all intermediate calculations, which can be found in the appendix.

3.1. One zero eigenvalue

Let us take the non-zero colour vector a^a as an eigenvector with a zero eigenvalue:

$$a^a K^{ab} = K^{ab}a^b = 0 \qquad a^2 = a^a a^a > 0. \tag{10}$$

Accordingly from the observation

$$\mathrm{Det}(K^{ab} + a^a a^b) \neq 0 \tag{11}$$

one may deduce that there is a matrix Δ^{ab}_1 with properties

$$K^{ab}\Delta^{bc}_1 = \Delta^{ab}_1 K^{bc} = \delta^{ac} - a^a a^c / a^2 \tag{12a}$$

$$a^a \Delta^{ab}_1 = \Delta^{ab}_1 a^b \equiv 0. \tag{12b}$$

Because the matrix $\mathbb{L}$ has the same non-zero eigenvalues as the matrix $\mathbb{K}$, one may find two orthogonal (Euclidean) eigenvectors n_μ and m_μ:

$$n_\mu L_{\mu\nu} = L_{\mu\nu} n_\nu = 0 \tag{13a}$$

$$m_\mu L_{\mu\nu} = L_{\mu\nu} m_\nu = 0 \tag{13b}$$

$$m_\mu n_\mu = 0 \tag{13c}$$

$$m^2 = m_\mu m_\mu > 0 \qquad n^2 = n_\mu n_\mu > 0.$$

Furthermore if one notices that

$$\mathrm{Det}(L_{\mu\nu} + n_\mu n_\nu + m_\mu m_\nu) \neq 0 \tag{14}$$

then one will obtain a matrix $D_{2\mu\nu}$ with properties:

$$L_{\mu\nu} D_{2\nu\lambda} = D_{2\mu\nu} L_{\nu\lambda} = \delta_{\mu\lambda} - n_\mu n_\lambda / n^2 - m_\mu m_\lambda / m^2 \tag{15a}$$

$$D_{2\mu\nu} n_\nu = n_\mu D_{2\mu\nu} \equiv 0 \tag{15b}$$

$$D_{2\mu\nu} m_\nu = m_\mu D_{2\mu\nu} \equiv 0. \tag{15c}$$

The degeneracy of external current prohibits us from transferring immediately the properties (10), (13a), (13b) into the appropriate relations for J^a_μ. However, if we restrict our discussion to real-valued objects, then by virtue of

$$a^a K^{ab} a^b = a^a J^a_\mu a^b J^b_\mu = 0$$

$$n_\mu L_{\mu\nu} n_\nu = n_\mu J^a_\mu n_\nu J^a_\nu = 0$$

$$m_\mu L_{\mu\nu} m_\nu = m_\mu J^a_\mu m_\nu J^a_\nu = 0$$

we will obtain the following properties of J^a_μ:

$$n_\mu J^a_\mu \equiv 0 \tag{16a}$$

$$m_\mu J^a_\mu \equiv 0 \tag{16b}$$

$$a^a J^a_\mu \equiv 0. \tag{16c}$$

Actually we may derive these equations after some manipulation of equations (10), (12a), (13a), (13b) and (15a) even for complex-valued objects (details can be found in the appendix).

So far our discussion has dealt with the structure of external current. We have shown that the actual class of J^a_μ may be equivalently defined either by equation (16) or by (10) and (13). We stress that these properties of J^a_μ (16) *should not be confused with constraints induced by an interaction with a gauge field system.*

Now we are in a position to solve the covariant conservation law (1) for the present case of J^a_μ. First we notice that the gauge field potential A^a_μ may be parametrised as

$$A^a_\mu = D_{2\mu\nu}\{S_1^{Tab} + (\delta^{ad} + a^a a^d / a^2)\varepsilon^{dbc} N^c\} J^b_\nu + \alpha^a m_\mu + \beta^a n_\mu \tag{17a}$$

where

$$S_1^{Tab} = (\delta^{ac} - a^a a^c / a^2) S^{cd} (\delta^{db} - a^d a^b / a^2) \tag{17b}$$

$$\alpha^a = A^a_\mu n_\mu / n^2 \tag{17c}$$

$$\beta^a = A^a_\mu m_\mu / m^2. \tag{17d}$$

Thus we may treat α^a, β^a, S_1^{Tab}, N^a as independent gauge field degrees of freedom. Accordingly from equation (2) we have again a constraint on gauge fields

$$N^a = -(1/2g)\partial_\mu J^a_\mu \tag{8}$$

and no restriction on external current.

3.2. *Two zero eigenvalues*

Here we will deal with another class of non-generic external current J^a_μ—when matrix $\mathbb{K}$ has two zero eigenvalues. Now we choose a^a and b^a as the appropriate orthogonal eigenvectors:

$$a^a K^{ab} = K^{ab} a^b = 0 \tag{18a}$$

$$b^a K^{ab} = K^{ab} b^b = 0 \tag{18b}$$

$$a^a b^b = 0 \tag{18c}$$

$$a^2 = a^a a^a > 0 \qquad b^2 = b^a b^a > 0.$$

Similarly to the previous case one proves that here a 'reciprocal' matrix Δ_2^{ab} has the following properties:

$$K^{ab}\Delta_2^{bc} = \Delta_2^{ab} K^{bc} = \delta^{ac} - a^a a^b / a^2 - b^a b^b / b^2 \tag{19a}$$

$$a^a \Delta_2^{ab} = \Delta_2^{ab} a^b \equiv 0 \tag{19b}$$

$$b^a \Delta_2^{ab} = \Delta_2^{ab} b^b \equiv 0. \tag{19c}$$

Besides, matrix $\mathbb{L}$ has three zero eigenvalues; thus there are three orthogonal eigenvectors n_μ, m_μ, p_μ

$$n_\mu L_{\mu\nu} = L_{\mu\nu} n_\nu = 0 \tag{20a}$$

$$m_\mu L_{\mu\nu} = L_{\mu\nu} m_\nu = 0 \tag{20b}$$

$$p_\mu L_{\mu\nu} = L_{\mu\nu} p_\nu = 0 \tag{20c}$$

$$m_\mu n_\mu = 0 \qquad p_\mu n_\mu = 0 \qquad m_\mu p_\mu = 0 \tag{20d}$$

$$m^2 = m_\mu m_\mu > 0 \qquad n^2 = n_\mu n_\mu > 0 \qquad p^2 = p_\mu p_\mu > 0.$$

With the help of these vectors one may find from the observation

$$\mathrm{Det}(L_{\mu\nu} + n_\mu n_\nu + m_\mu m_\nu + p_\mu p_\nu) \neq 0$$

that there exists a matrix $D_{3\mu\nu}$ which satisfies the following properties:

$$L_{\mu\nu} D_{3\nu\lambda} = D_{3\mu\nu} L_{\nu\lambda} = \delta_{\mu\lambda} - n_\mu n_\lambda / n^2 - m_\mu m_\lambda / m^2 - p_\mu p_\lambda / p^2 \tag{21a}$$

$$D_{3\mu\nu} n_\nu = n_\mu D_{3\mu\nu} \equiv 0 \tag{21b}$$

$$D_{3\mu\nu} m_\nu = m_\mu D_{3\mu\nu} \equiv 0 \tag{21c}$$

$$D_{3\mu\nu} p_\nu = p_\mu D_{3\mu\nu} \equiv 0. \tag{21d}$$

Furthermore, from the properties of the matrices $\mathbb{K}$ and $\mathbb{L}$ we want to extract appropriate relations for J^a_μ. For a real-valued external current J^a_μ one may easily arrive at

$$n_\mu J^a_\mu = m_\mu J^a_\mu = p_\mu J^a_\mu \equiv 0 \tag{22a}$$

$$a^a J^a_\mu = b^a J^a_\mu \equiv 0. \tag{22b}$$

At present we do not have proof that these properties can be obtained for a complex-valued J^a_μ; we leave this as an open question.

In order to solve the covariant conservation law (2) we will introduce the following parametrisation of the gauge field potential A^a_μ:

$$A^a_\mu = D_{3\mu\nu}\{S^{Tab}_2 + (\delta^{ad} + a^a a^d/a^2 + b^a b^d/b^2)\varepsilon^{dbc}N^c\}J^b_\mu + \alpha^a m_\mu + \beta^a n_\mu + \gamma^a p^a \tag{23a}$$

where

$$S^{Tab}_2 = (\delta^{ac} - a^a a^c/a^2 - b^a b^c/b^2)S^{cd}(\delta^{db} - a^d a^b/a^2 - b^d b^b/b^2) \tag{23b}$$

$$N^{La} = a^b N^b/a^2 a^a + b^b N^b/b^2 b^a = n_1 a^a + n_2 b^a \tag{23c}$$

$$\alpha^a = A^a_\mu n_\mu/n^2 \tag{23d}$$

$$\beta^a = A^a_\mu m_\mu/m^2 \tag{23e}$$

$$\gamma^a = A^a_\mu p_\mu/p^2 p \tag{23f}$$

and treat α^a, β^a, γ^a, S^{Tab}_2, n_1, n_2 as independent gauge field degrees of freedom. Looking at (2) one finds that n_1 and n_2 are dynamically constrained:

$$n_1 = -(1/2ga^2)\partial_\mu J^a_\mu a^a \tag{24a}$$

$$n_2 = -(1/2gb^2)\partial_\mu J^a_\mu b^a \tag{24b}$$

and that there is also a constraint on J^a_μ

$$\partial_\mu J^a_\mu \varepsilon^{abc} a^b b^c = 0. \tag{24c}$$

4. Conclusions

Finally one arrives at the apparently surprising conclusion that for arbitrary non-Abelian external current the covariant conservation law imposes constraints on gauge field degrees of freedom. External current J^a_μ may be called intrinsically non-Abelian if the matrix $\mathbb{K}$ (4a) has at least two non-zero eigenvalues. Otherwise J^a_μ can be called Abelian (Jackiw and Rossi 1980, Kiskis 1980, Weiss 1980) because it factorises by means of a suitable gauge transformation

$$J^a_\mu(x) = \lambda^a(x) j_\mu(x) \qquad \lambda^a(x)\lambda^a(x) = 1. \tag{25}$$

Morever if the colour vector λ^a is a smooth function then its spacetime dependence may be gauge transformed by a non-singular $\mathbb{R}$ matrix

$$\lambda^a(x) = R^{ab}(x)\bar{\lambda}^b(x_0) \tag{26a}$$

where

$$R^{ba}R^{bc} = \delta^{ac} \qquad R^{am}R^{bn}R^{cp}\varepsilon^{abc} = \varepsilon^{mnp}.$$

With the same matrix $\mathbb{R}$ one may gauge transform the gauge field potential A^a_μ

$$\bar{A}^a_\mu(x) = R^{ba}(x)A^b_\mu(x) - (1/2g)\varepsilon^{abc}R^{db}(x)\partial_\mu R^{dc}(x). \tag{26b}$$

In terms of these new variables equation (2) may be rewritten as

$$\partial_\mu j_\mu = 0 \tag{27a}$$

$$j_\mu \bar{A}^b_\mu(\delta^{ab}\lambda^c\lambda^c - \lambda^a\lambda^b) = 0. \tag{27b}$$

The above observation shows explicitly that the covariant conservation law (2) produces a true constraint on the external current only when this current is Abelian.

Overall this allows the general conclusion that if one tests non-Abelian problems (like equation (2)) with Abelian objects (like (25)) then *some singularities may arise and final results are not representative for generic non-Abelian objects.*

Finally, we would like to mention that other conservation laws may have a different interpretation. If both entities are dynamical, then the final result will depend on the actual physical content. For example the SU(N) non-Abelian Gauss law has been solved as a constraint on gauge field momenta E_i^a keeping gauge field potentials A_i^a arbitrary (Baluni and Grossman 1978, Das *et al* 1979, Goldstone and Jackiw 1978, Izergin *et al* 1979).

Appendix

Here we present essential details of calculations, which have been omitted in the main part of the paper. First we notice that for any symmetric 4×4 matrix $\mathbb{X}$

$$\mathrm{Det}\,\mathbb{X} \equiv \tfrac{1}{24}(\mathrm{Tr}\,\mathbb{X})^4 - \tfrac{1}{4}(\mathrm{Tr}\,\mathbb{X}^4) + \tfrac{1}{3}(\mathrm{Tr}\,\mathbb{X})(\mathrm{Tr}\,\mathbb{X}^3) + \tfrac{1}{8}(\mathrm{Tr}\,\mathbb{X}^2)^2 - \tfrac{1}{4}(\mathrm{Tr}\,\mathbb{X}^2)(\mathrm{Tr}\,\mathbb{X})^2 \qquad (\mathrm{A}1a)$$

and for any symmetric 3×3 matrix $\mathbb{Y}$

$$\mathrm{Det}\,\mathbb{Y} \equiv \tfrac{1}{6}(\mathrm{Tr}\,\mathbb{Y})^3 + \tfrac{1}{3}(\mathrm{Tr}\,\mathbb{Y}^3) - \tfrac{1}{2}(\mathrm{Tr}\,\mathbb{Y})(\mathrm{Tr}\,\mathbb{Y}^2). \qquad (\mathrm{A}1b)$$

Because Det and Tr operations are invariant for a diagonalisation procedure, then the above equations can be easily proved by a direct calculation if matrices $\mathbb{X}$ and $\mathbb{Y}$ are diagonal. Secondly from the definitions (4a) and (4b) we have equality of traces for matrices $\mathbb{K}$ and $\mathbb{L}$

$$\mathrm{Tr}(\mathbb{L}^n) = \mathrm{Tr}(\mathbb{K}^n) \qquad n = 1, \ldots. \qquad (\mathrm{A}2)$$

Now we may take the case of generic external current when equations (6a) and (6b) are fulfilled. Let us consider a new Euclidean matrix $L_{\mu\nu} + n_\mu n_\nu$ which is already non-singular:

$$\mathrm{Det}(L_{\mu\nu} + n_\mu n_\nu) = n^2\,\mathrm{Det}(K^{ab}) \neq 0 \qquad (\mathrm{A}3)$$

because, due to (6b) and (A2), we have

$$\begin{aligned}
&\mathrm{Det}(L_{\mu\nu} + n_\mu n_\nu) \\
&\quad = \tfrac{1}{24}[\mathrm{Tr}\,\mathbb{L} + n^2]^4 - \tfrac{1}{4}[\mathrm{Tr}\,\mathbb{L}^4 + (n^2)^4] + \tfrac{1}{3}[\mathrm{Tr}\,\mathbb{L} + n^2][\mathrm{Tr}\,\mathbb{L}^3 + (n^2)^3] \\
&\qquad + \tfrac{1}{8}[\mathrm{Tr}\,\mathbb{L}^2 + (n)^2]^2 - \tfrac{1}{4}[\mathrm{Tr}\,\mathbb{L}^2 + (n^2)^2][\mathrm{Tr}\,\mathbb{L} + n^2]^2 \\
&\quad = \{\tfrac{1}{24}(\mathrm{Tr}\,\mathbb{K})^4 - \tfrac{1}{4}(\mathrm{Tr}\,\mathbb{K}^4) + \tfrac{1}{3}(\mathrm{Tr}\,\mathbb{K})(\mathrm{Tr}\,\mathbb{K}^3) + \tfrac{1}{8}(\mathrm{Tr}\,\mathbb{K}^2)^2 - \tfrac{1}{4}(\mathrm{Tr}\,\mathbb{K}^2)(\mathrm{Tr}\,\mathbb{K})^2\} \\
&\qquad + \{\tfrac{1}{6}(\mathrm{Tr}\,\mathbb{K})^3 + \tfrac{1}{3}(\mathrm{Tr}\,\mathbb{K}^3) - \tfrac{1}{2}(\mathrm{Tr}\,\mathbb{K})(\mathrm{Tr}\,\mathbb{K}^2)\}n^2.
\end{aligned}$$

According to (A1a) we find that the first curly bracket vanishes identically as a four-dimensional determinant of a three-dimensional matrix. Yet from (A1b) we learn that the second curly bracket gives a three-dimensional determinant of $\mathbb{K}$ which, by assumption, is different from zero.

Furthermore, due to the non-singularity of matrix $L_{\mu\nu} + n_\mu n_\nu$ we may define a symmetric matrix $D^T_{1\mu\nu}$ by the reciprocity relations

$$(L_{\mu\nu} + n_\mu n_\nu/n^2)(D^T_{1\nu\rho} + n_\nu n_\rho/n^2) = \delta_{\mu\rho} \qquad (\mathrm{A}4a)$$

$$(D^T_{1\mu\nu} + n_\mu n_\nu/n^2)(L_{\nu\rho} + n_\nu n_\rho/n^2) = \delta_{\mu\rho}. \qquad (\mathrm{A}4b)$$

Let us multiply the first equation by n_μ and the second one by n_ν; then according to (6b) we arrive at

$$n_\nu D^T_{1\nu\mu} + n_\mu = n_\mu \qquad D^T_{1\nu\mu} n_\mu + n_\nu = n_\nu .$$

Thus we have following properties of $D^T_{1\mu\nu}$:

$$n_\mu D^T_{1\mu\nu} = D^T_{1\mu\nu} n_\nu = 0 \tag{A5}$$

and equations (A4a) and (A4b) are equivalent to

$$D^T_{1\mu\nu} L_{\nu\rho} = L_{\mu\nu} D^T_{1\nu\rho} = \delta_{\mu\rho} - n_\mu n_\rho / n^2 . \tag{7b}$$

We clearly see that the matrix $D^T_{1\mu\nu}$ exists as long as equation (A3) is satisfied.

Finally one finds a decomposition of arbitrary A^a_μ:

$$\begin{aligned} A^a_\mu &= A^a_\nu n_\nu n_\mu / n^2 + (\delta_{\mu\nu} - n_\mu n_\nu / n^2) A^a_\nu \\ &= A^a_\nu n_\nu n_\mu / n^2 + D^T_{1\mu\nu} L_{\nu\rho} A^a_\rho \\ &= \alpha^a n_\mu + D^T_{1\mu\nu} J^b_\nu J^b_\rho A^a_\rho \\ &= \alpha^a n_\mu + D^T_{1\mu\nu} (S^{ab} + \varepsilon^{abc} N^c) J^b_\nu \end{aligned}$$

where S^{ab}, N^a, α^a are given by (7b), (7c) and (7d) respectively. This completes our calculations in the generic case and we turn to a degenerate one.

First we suppose that matrix $\mathbb{K}$ has one zero eigenvalue (10); thus we can find

$$\begin{aligned} \mathrm{Det}(K^{ab} + a^a a^b) &= \tfrac{1}{6}[\mathrm{Tr}\,\mathbb{K} + a^2]^3 + \tfrac{1}{3}[\mathrm{Tr}\,\mathbb{K}^3 + (a^2)^3] - \tfrac{1}{2}[\mathrm{Tr}\,\mathbb{K} + a^2][\mathrm{Tr}\,\mathbb{K}^2 + (a^2)^2] \\ &= \{\tfrac{1}{6}(\mathrm{Tr}\,\mathbb{K})^3 + \tfrac{1}{3}(\mathrm{Tr}\,\mathbb{K}^3) - \tfrac{1}{2}(\mathrm{Tr}\,\mathbb{K})(\mathrm{Tr}\,\mathbb{K}^2)\} + \tfrac{1}{2}a^2\{(\mathrm{Tr}\,\mathbb{K})^2 - \mathrm{Tr}\,\mathbb{K}^2\} \neq 0. \end{aligned} \tag{A6}$$

The first curly bracket vanishes because it is equal to $\mathrm{Det}\,\mathbb{K}$, while the second one is non-zero if there are at least two non-zero eigenvalues. Thus one may define matrix Δ^{ab}_1 by the reciprocity condition

$$(K^{ab} + a^a a^b / a^2)(\Delta^{bc}_1 + a^b a^c / a^2) = (\Delta^{ab}_1 + a^a a^b / a^2)(K^{bc} + a^b a^c / a^2) = \delta^{ac} \tag{A7}$$

and one can easily derive the following properties of Δ^{ab}_1:

$$K^{ab} \Delta^{bc}_1 = \Delta^{ab}_1 K^{bc} = \delta^{ac} - a^a a^c / a^2 \tag{12a}$$

$$a^a \Delta^{ab}_1 = \Delta^{ab}_1 a^b \equiv 0. \tag{12b}$$

Furthermore, from equations (13) and (A1) one may prove that

$$\mathrm{Det}(L_{\mu\nu} + m_\mu m_\nu + n_\mu n_\nu) = \tfrac{1}{2} n^2 m^2 [(\mathrm{Tr}\,\mathbb{K})^2 - \mathrm{Tr}\,\mathbb{K}^2] \neq 0. \tag{A8}$$

Thus if one defines matrix $D_{2\mu\nu}$ by another reciprocity condition

$$\begin{aligned} &(L_{\mu\nu} + n_\mu n_\nu / n^2 + m_\mu m_\nu / m^2)(D^T_{1\nu\rho} + n_\nu n_\rho / n^2 + m_\nu m_\rho / m^2) \\ &\quad = (D^T_{1\mu\nu} + n_\mu n_\nu / n^2 + m_\mu m_\rho / m^2)(L_{\nu\rho} + n_\nu n_\rho / n^2 + m_\nu m_\rho / m^2) = \delta_{\mu\rho} \end{aligned} \tag{A9a}$$

then one will easily arrive at the properties (15a–c).

Up until now the actual analysis has been performed along similar lines to the analysis for generic external current. However, if one wants to find properties of external current then one encounters obstacles connected with the degeneracy of J^a_μ. First from (10), (13a) and (13b), one finds

$$a^a K^{ab} J^b_\nu = a^a J^a_\mu L_{\mu\nu} = 0 \tag{A9b}$$

$$n_\nu L_{\nu\mu} J^a_\mu = n_\nu J^b_\nu K^{ba} = 0 \tag{A9c}$$

$$m_\nu L_{\nu\mu} J^a_\mu = m_\nu J^b_\nu K^{ba} = 0. \tag{A9d}$$

In order to get rid of the matrices $\mathbb{K}$ and $\mathbb{L}$ one can apply the previously defined matrices $D_{2\mu\nu}$ (13*a*) and Δ_1^{ab} (12*a*):

$$a^a J^a_\mu = a^a J^a_\nu m_\nu m_\mu / m^2 + a^a J^a_\nu n_\nu n_\mu / n^2 \tag{A10a}$$

$$n_\mu J^a_\mu = n_\mu J^b_\mu a^b a^a / a^2 \tag{A10b}$$

$$m_\mu J^a_\mu = m_\mu J^b_\mu a^b a^a / a^2. \tag{A10c}$$

Furthermore, from (A10*b*, *c*) one easily obtains

$$n_\mu L_{\mu\nu} = n_\mu J^a_\mu J^a_\nu = n_\mu J^b_\mu a^b a^c J^c_\nu / a^2 = 0 \tag{A11a}$$

$$m_\mu L_{\mu\nu} = m_\mu J^a_\mu J^a_\nu = m_\mu J^b_\mu a^b a^c J^c_\nu / a^2 = 0. \tag{A11b}$$

Thus there are two possibilities: either

$$a^a J^a_\mu = 0 \tag{A12a}$$

or

$$n_\mu J^b_\mu a^b = m_\mu J^b_\mu a^b = 0. \tag{A12b}$$

The first alternative, due to (A10*b*) and (A10*c*), immediately leads to the conclusion that equation (A12*b*) must also be satisfied. The second one, due to relation (A10*a*), gives (A12*a*). Thus we have proved that even for a complex-valued external current the properties (A12*a*) and (A12*b*) are satisfied.

So far the analysis has concerned the structure of J^a_μ; now we should turn to the covariant conservation law. In order to find a suitable parametrisation of the gauge field potential A^a_μ let us study S^{ab} and N^a defined by (7*b*) and (7*c*). Due to the property (16*c*) we have

$$A^a_\mu J^b_\mu a^b = (S^{ab} + \varepsilon^{abc} N^c) a^b = 0 \tag{A13a}$$

$$a^a A^b_\mu J^a_\mu = a^a (S^{ab} - \varepsilon^{abc} N^c) = 0 \tag{A13b}$$

or, equivalently,

$$S^{ab} a^b = -\varepsilon^{abc} a^b N^c \tag{A13a'}$$

$$a^a S^{ab} = \varepsilon^{abc} a^b N^c. \tag{A13b'}$$

Thus we see that S^{ab} and N^a are not independent and only certain components of them can be taken as parameters of A^a_μ. It is useful to take the transversal matrix S_1^{Tab}

$$S_1^{Tab} = (\delta^{ac} - a^a a^c / a^2) S^{cd} (\delta^{db} - a^d a^b / a^2) \tag{17b}$$

and N^a as linearly independent quantities. The matrix S^{ab} may be expressed as

$$\begin{aligned} S^{ab} &= (\delta^{ac} - a^a a^c / a^2) S^{cd} (\delta^{db} - a^d a^b / a^2) + a^a a^c S^{cb} / a^2 + a^b a^d S^{ad} / a^2 \\ &\quad - a^a a^b / a^2 a^c S^{cd} a^d / a^2 \\ &= S_1^{Tab} - a^a \varepsilon^{bcd} a^c N^d / a^2 - a^b \varepsilon^{acd} a^c N^d / a^2 \end{aligned} \tag{A14}$$

and accordingly

$$(S^{ab} + \varepsilon^{abc} N^c) J^b_\nu = \{ S_1^{Tab} + (\delta^{ad} + a^a a^d / a^2) \varepsilon^{dbc} N^c \} J^b_\nu. \tag{A15}$$

Now we are in a position to write down a proper decomposition of A^a_μ:

$$\begin{aligned} A^a_\mu &= (\delta_{\mu\nu} - m_\mu m_\nu / m^2 - n_\mu n_\nu / n^2) A^a_\nu + A^a_\nu n_\nu / n^2 n_\nu + A^a_\nu m_\nu / m^2 m_\mu \\ &= D_{2\mu\nu} J^b_\nu J^b_\lambda A^a_\lambda + \alpha^a n_\mu + \beta^a m_\mu \\ &= D_{2\mu\nu} (S^{ab} + \varepsilon^{abc} N^c) J^b_\nu + \alpha^a n_\mu + \beta^a m_\mu \\ &= \alpha^a m_\mu + \beta^a n_\mu + D_{2\mu\nu} \{ S_1^{Tab} + (\delta^{ad} + a^a a^d / a^2) \varepsilon^{dbc} N^c \} J^b_\nu \end{aligned} \tag{A16}$$

where α^a, β^a, are given by (17*b*) and (17*c*), respectively. This completes our calculations concerning the degenerate case of J^a_μ, when the matrix $\mathbb{K}$ has one zero eigenvalue. We would like to stress that S^{ab} and N^a are mutually constrained and this phenomenon appears in both classes of degenerate external current.

Now we turn to the last case when the matrix $\mathbb{K}$ has two zero eigenvalues. First, from the formulae (A1*a*) and (A1*b*), we find that

$$\mathrm{Det}(K^{ab}+a^a a^b+b^a b^b)=a^2b^2\,\mathrm{Tr}\,\mathbb{K}\neq 0 \qquad (\mathrm{A}17a)$$

$$\mathrm{Det}(L_{\mu\nu}+n_\mu n_\nu+m_\mu m_\nu+p_\mu p_\nu)=n^2m^2p^2\,\mathrm{Tr}\,\mathbb{K}\neq 0. \qquad (\mathrm{A}17b)$$

Furthermore, we may define matrices $\Delta_{3\mu\nu}$ and D_2^{ab} by the reciprocity relations

$$\begin{aligned}(L_{\mu\nu}+n_\mu n_\nu/n^2+m_\mu m_\nu/m^2+p_\mu p_\nu/p^2)(D^T_{1\nu\rho}+n_\nu n_\rho/n^2+m_\nu m_\rho/m^2+p_\nu p_\rho/p^2)\\ =(D^T_{1\mu\nu}+n_\mu n_\nu/n^2+m_\mu m_\nu/m^2+p_\mu p_\nu/p^2)\\ \times(L_{\nu\rho}+n_\nu n_\rho/n^2+m_\nu m_\rho/m^2+p_\nu p_\rho/p^2)=\delta_{\mu\rho}\end{aligned} \qquad (\mathrm{A}18a)$$

$$\begin{aligned}(K^{ab}+a^a a^b/a^2+b^a b^b/b^2)(\Delta_1^{bc}+a^b a^c/a^2+b^b b^c/b^2)\\ =(\Delta_1^{ab}+a^a a^b/a^2+b^a b^b/b^2)(K^{bc}+a^b a^c/a^2+b^b b^c/b^2)=\delta^{ac}.\end{aligned} \qquad (\mathrm{A}18b)$$

Proceeding in a similar manner to the previous case one may easily derive properties (19*b*), (19*c*) and (21*a*–*c*). However, if one seeks the properties of J^a_μ then one cannot imitate the steps from (A9) to (A12). Currently we must restrict the possible J^a_μ to a real-valued function. Thus we take the obvious identities

$$a^aK^{ab}a^b=a^aJ^a_\mu a^bJ^b_\mu\equiv 0 \qquad (\mathrm{A}19a)$$

$$b^aK^{ab}b^b=b^aJ^a_\mu b^bJ^b_\mu\equiv 0 \qquad (\mathrm{A}19b)$$

$$n_\mu L_{\mu\nu}n_\nu=n_\mu J^a_\mu n_\nu J^a_\nu\equiv 0 \qquad (\mathrm{A}19c)$$

$$m_\mu L_{\mu\nu}m_\nu=m_\mu J^a_\mu m_\nu J^a_\nu\equiv 0 \qquad (\mathrm{A}19d)$$

$$p_\mu L_{\mu\nu}p_\nu=p_\mu J^a_\mu p_\nu J^a_\nu\equiv 0 \qquad (\mathrm{A}19e)$$

and for real Euclidean vectors we conclude that J^a_μ has the following properties:

$$n_\mu J^a_\mu=m_\mu J^a_\mu=p_\mu J^a_\mu\equiv 0 \qquad (22a)$$

$$a^aJ^a_\mu=b^aJ^a_\mu\equiv 0. \qquad (22b)$$

Finally we would like to find a parametrisation of the gauge field A^a_μ that is suitable for solving equation (2). We notice that due to (22*b*) there are two basic constraints:

$$A^a_\mu J^b_\mu a^b=(S^{ab}+\varepsilon^{abc}N^c)a^b=0 \qquad (\mathrm{A}20a)$$

$$A^a_\mu J^b_\mu b^b=(S^{ab}+\varepsilon^{abc}N^c)b^b=0. \qquad (\mathrm{A}20b)$$

Because matrix S^{ab} is symmetric

$$a^aS^{ab}b^b=b^aS^{ab}a^b \qquad (\mathrm{A}20c)$$

then from (A20*a*) and (A20*b*) we have

$$\varepsilon^{abc}a^ab^bN^c\equiv 0. \qquad (\mathrm{A}20d)$$

For these observations one may deduce which parts of S^{ab} and N^a are not mutually constrained. First we notice that the longitudinal part of N^a

$$N^{La}=a^bN^b/a^2a^a+b^bN^b/b^2b^a=n_1a^a+n_2b^a \qquad (23c)$$

carries two degrees of freedom. Furthermore, by virtue of the above constraints one may choose a missing degree of freedom. We want to take S_2^{Tab} defined by (23b); however, another choice is equally possible. Here we decompose S^{ab} in terms of S_2^{Tab} and N^{La}:

$$\begin{aligned} S^{ab} &= (\delta^{ac} - a^a a^c/a^2 - b^a b^c/b^2) S^{cd} (\delta^{db} - a^d a^b/a^2 - b^d b^b/b^2) + a^a a^c S^{cb}/a^2 \\ &\quad + b^a b^c S^{cb}/b^2 + a^b a^d S^{ad}/a^2 + b^b b^d S^{ad}/b^2 \\ &\quad - a^a b^b/a^2 a^c S^{cd} b^d/b^2 - b^a a^b/a^2 b^c S^{cd} a^d/b^2 \\ &= S_2^{Tab} - (a^a \varepsilon^{bcd} + a^b \varepsilon^{acd}) a^c N^{Ld}/a^2 - (b^a \varepsilon^{bcd} + b^b \varepsilon^{acd}) b^c N^{Ld}/b^2 \end{aligned} \tag{A21}$$

and we write

$$(S^{ab} + \varepsilon^{abc} N^c) J_\nu^b = S_2^{Tab} J_\nu^b + (\delta^{ad} + a^a a^d/a^2 + b^a b^d/b^2) \varepsilon^{dbc} N^c J_\nu^b. \tag{A22}$$

This allows us to make the following decomposition of any A_ν^b:

$$\begin{aligned} A_\mu^a &= (\delta_{\mu\nu} - m_\mu m_\nu/m^2 - n_\mu n_\nu/n^2 - p_\mu p_\nu/p^2) A_\nu^a + A_\nu^a n_\nu/n^2 n_\mu \\ &\quad + A_\nu^a m_\nu/m^2 m_\mu + A_\nu^a p_\nu/p^2 p_\mu \\ &= D_{3\mu\nu} J_\nu^b J_\lambda^b A_\lambda^a + \alpha^a n_\mu + \beta^a m_\mu + \gamma^a p_\mu \\ &= D_{3\mu\nu} (S^{ab} + \varepsilon^{abc} N^c) J_\nu^b + \alpha^a n_\mu + \beta^a m_\mu + \gamma^a p_\mu \\ &= D_{3\mu\nu} \{ S_2^{Tab} + (\delta^{ad} + a^a a^d/a^2 + b^a b^d/b^2) \varepsilon^{dbc} N^c \} J_\nu^b + \alpha^a m_\mu + \beta^a n_\mu + \gamma^a p_\mu . \end{aligned}$$

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